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UNIT 5

SL 5.4 — Tangents and Normals

What is a Tangent • Slope = f′(a) • Tangent Lines • Normal Lines • Horizontal Tangents • Using Technology

A. KEY VOCABULARY & RULES (FILL IN THE BLANKS)

These are the ideas behind today's lesson on tangents and normals.

1. A **tangent** to a curve at a point is a straight line that touches the curve there and has the same _____ as the curve at that point.
2. The slope of the tangent to $y = f(x)$ at $x = a$ is _____.
3. Equation of the tangent (point-slope form): $y - f(a) =$ _____.
4. Two **perpendicular** lines have slopes whose product is _____.
5. If the tangent has slope m, the **normal** has slope $-\frac{1}{m}$ = _____.
6. Equation of the normal at $x = a$: $y - f(a) =$ _____.

B. WORKED METHOD — FINDING A TANGENT

Step 1: find $f(a)$. Step 2: find $f'(a)$. Step 3: use $y - f(a) = f'(a)(x - a)$.

1. Find the tangent to $f(x) = 3x^2 + 2x + 5$ at $x = 1$.
   - (a) $f(1) =$ _____
   - (b) $f'(x) =$ _____
   - (c) $f'(1) =$ _____
   - (d) Equation of the tangent: _____

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SL 5.4 — Tangents and Normals

Finding Tangent Lines • Finding Normal Lines

C. FINDING TANGENT LINES

Give each answer in the form $y = mx + c$.

1. $y = x^2$ **at** $x = 2$: _____
2. $y = x^2 + 3x$ **at** $x = 1$: _____
3. $y = x^3 - 2x$ **at** $x = 2$: _____
4. $y = 2x^2 - 5x + 1$ **at** $x = 0$: _____

D. FINDING NORMAL LINES

The normal slope is $-\frac{1}{f'(a)}$. Give each answer in point-slope form.

1. $y = x^2 + 3x$ **at** $x = 1$: _____
2. $y = x^2$ **at** $x = 2$: _____
3. $y = x^3$ **at** $x = 1$: _____

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UNIT 5

SL 5.4 — Tangents and Normals

Tangent & Normal Together • Horizontal Tangents

E. TANGENT AND NORMAL TOGETHER

Find the point and gradient first, then write both lines.

1. Let $f(x) = x^2 - 4x + 5$.

(a) Find $f(3)$ and $f'(3)$. _____

(b) Equation of the tangent at $x = 3$. _____

(c) Equation of the normal at $x = 3$. _____

F. HORIZONTAL TANGENTS

A tangent is horizontal where $f'(x) = 0$.

1. $f(x) = x^2 - 6x + 5$. Find where the tangent is horizontal and give its equation.

Answer: _____

2. $f(x) = x^3 - 3x$. Find the x-values where the tangent is horizontal.

Answer: _____

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UNIT 5

SL 5.4 — Tangents and Normals

Extended: Tangent & Normal Intercepts

G. EXTENDED: TANGENT LINE & INTERCEPTS

Work with the tangent as a full straight-line equation.

1. Consider $y = x^2 - 2x$ at $x = 3$.

(a) Find the tangent in the form $y = mx + c$. _____

(b) Find where this tangent crosses the x-axis. _____

(c) Find where this tangent crosses the y-axis. _____

H. EXTENDED: NORMAL LINE & INTERCEPTS

The normal is perpendicular to the tangent at the point of contact.

1. Consider $y = x^2$ at $x = 1$.

(a) Find the equation of the normal in the form $y = mx + c$. _____

(b) Find where the normal meets the y-axis. _____

(c) Find where the normal meets the x-axis. _____

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UNIT 5

SL 5.4 — Tangents and Normals

Extended: Work Backwards • Parallel & Perpendicular

I. EXTENDED: WORK BACKWARDS (FIND THE UNKNOWN)

Use the gradient condition to build an equation and solve.

1. **The tangent to $y = x^2 + bx$ at $x = 2$ has gradient 7. Find b.** _____

2. **The curve $y = x^2 + kx + 1$ has a horizontal tangent at $x = 2$. Find k.**

   _____

3. **The tangent to $y = ax^2$ at $x = 1$ has gradient 6. Find a.** _____

J. EXTENDED: PARALLEL & PERPENDICULAR TANGENTS

Parallel lines share a gradient; perpendicular gradients multiply to -1.

1. **Find the point on $y = x^2$ where the tangent is parallel to $y = 6x - 1$.**

   _____

2. **Find the point on $y = x^2$ where the tangent is perpendicular to $y = -\frac{1}{4}x$.**

   _____

3. **Find the equation of the tangent to $y = x^2 - x$ that is parallel to $y = 3x$.**

   _____

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UNIT 5

SL 5.4 — Tangents and Normals

Extended: Modelling • Non-routine Reasoning

K. EXTENDED: MODELLING WITH TANGENTS

The gradient of the tangent is the instantaneous rate of change.

1. **The cross-section of a hill (metres) is modelled by $h(x) = -0.5x^2 + 4x$.**

   (a) Find the gradient of the hill at $x = 2$. _____

   (b) Find the equation of the tangent line at $x = 2$. _____

   (c) A ball leaves the hill along this tangent. Find its height on the line at $x = 5$. _____

L. NON-ROUTINE REASONING (HARD)

General tangents and points that satisfy extra conditions.

1. **Show that the tangent to $y = x^2$ at $x = a$ is $y = 2ax - a^2$.**

   _____

2. **The tangent to $y = x^2$ at a point P passes through $(0, -1)$. Find the coordinates of P.** _____

3. **The normal to $y = x^2$ at $x = 1$ meets the y-axis at Q. Find Q.**

   _____

ANSWER KEY

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UNIT 5

ANSWER KEY — SL 5.4 Tangents and Normals

Teachers: equivalent forms and explanations earn full credit.

A. KEY VOCABULARY & RULES

1. **slope (gradient).**
2. slope $= f'(a)$
3. $y - f(a) = f'(a)(x - a)$
4. -1
5. $-\frac{1}{m} = -1/m$ (negative reciprocal)
6. $y - f(a) = y - f(a) = -\frac{1}{f'(a)}(x - a)$

B. WORKED METHOD — FINDING A TANGENT

1. (a) $f(1) = 3(1)^2 + 2(1) + 5 = 10$
   (b) $f'(x) = 6x + 2$
   (c) $f'(1) = 6(1) + 2 = 8$
   (d) $y - 10 = 8(x - 1)$, **i.e.** $y = 8x + 2$

C. FINDING TANGENT LINES

1. $f'(x) = 2x$, $f'(2) = 4$, $f(2) = 4$: $y = 4x - 4$
2. $f'(x) = 2x + 3$, $f'(1) = 5$, $f(1) = 4$: $y = 5x - 1$
3. $f'(x) = 3x^2 - 2$, $f'(2) = 10$, $f(2) = 4$: $y = 10x - 16$
4. $f'(x) = 4x - 5$, $f'(0) = -5$, $f(0) = 1$: $y = -5x + 1$

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ANSWER KEY — SL 5.4 Tangents and Normals

Teachers: accept equivalent working and rounding.

D. FINDING NORMAL LINES

1. $f'(1) = 5$, **normal slope** $-\frac{1}{5}$, **at** $(1, 4)$: $y - 4 = -\frac{1}{5}(x - 1)$
2. $f'(2) = 4$, **normal slope** $-\frac{1}{4}$, **at** $(2, 4)$: $y - 4 = -\frac{1}{4}(x - 2)$
3. $f'(x) = 3x^2$, $f'(1) = 3$, **normal slope** $-\frac{1}{3}$, **at** $(1, 1)$: $y - 1 = -\frac{1}{3}(x - 1)$

E. TANGENT AND NORMAL TOGETHER

1. (a) $f(3) = 9 - 12 + 5 = 2$; $f'(x) = 2x - 4$, $f'(3) = 2$
   (b) Tangent: $y - 2 = 2(x - 3)$, **i.e.** $y = 2x - 4$
   (c) Normal: $y - 2 = -\frac{1}{2}(x - 3)$

F. HORIZONTAL TANGENTS

1. $f'(x) = 2x - 6 = 0 \Rightarrow x = 3$; $f(3) = -4$, **tangent** $y = -4$.
2. $f'(x) = 3x^2 - 3 = 0 \Rightarrow x = \pm 1$

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ANSWER KEY — SL 5.4 Tangents and Normals

Teachers: equivalent reasoning earns full credit.

G. TANGENT LINE & INTERCEPTS

1. (a) $f'(x) = 2x - 2$, $f'(3) = 4$, $f(3) = 3$: $y = 4x - 9$
   (b) x-axis: $0 = 4x - 9 \Rightarrow x = \frac{9}{4}$
   (c) y-axis: $x = 0 \Rightarrow y = -9$

H. NORMAL LINE & INTERCEPTS

1. (a) $f'(1) = 2$, **normal slope** $-\frac{1}{2}$, **at** $(1, 1)$: $y = -\frac{1}{2}x + \frac{3}{2}$
   (b) y-axis: $\left(0, \frac{3}{2}\right)$
   (c) x-axis: $y = 0 \Rightarrow x = 3$, **i.e.** $(3, 0)$

I. WORK BACKWARDS

1. $f'(x) = 2x + b$, $f'(2) = 4 + b = 7 \Rightarrow b = 3$
2. $f'(x) = 2x + k = 0$ **at** $x = 2 \Rightarrow k = -4$
3. $f'(x) = 2ax$, $f'(1) = 2a = 6 \Rightarrow a = 3$

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UNIT 5

ANSWER KEY — SL 5.4 Tangents and Normals

Teachers: accept equivalent forms and sensible rounding.

J. PARALLEL & PERPENDICULAR TANGENTS

1. $2x = 6 \Rightarrow x = 3$, **point** $(3, 9)$
2. **line slope** $-\frac{1}{4}$**, so tangent slope** $= 4$**:** $2x = 4 \Rightarrow x = 2$**, point** $(2, 4)$
3. $2x - 1 = 3 \Rightarrow x = 2$**,** $f(2) = 2$**:** $y = 3x - 4$

K. MODELLING WITH TANGENTS

1. (a) $h'(x) = -x + 4$**,** $h'(2) = 2$
   (b) $h(2) = 6$**:** $y - 6 = 2(x - 2)$**, i.e.** $y = 2x + 2$
   (c) **At** $x = 5$**:** $y = 2(5) + 2 = 12$ **m**

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UNIT 5

ANSWER KEY — SL 5.4 Tangents and Normals

Teachers: accept equivalent forms and sensible rounding.

L. NON-ROUTINE REASONING

1. $f'(x) = 2x$, **slope** $= 2a$, **point** (a, a^2): $y - a^2 = 2a(x - a) \Rightarrow y = 2ax - a^2$
2. **Tangent** $y = 2ax - a^2$ **through** $(0, -1)$: $-1 = -a^2 \Rightarrow a = \pm 1$; **so** $P = (1, 1)$ **or** $(-1, 1)$
3. **Normal slope** $-\frac{1}{2}$, **at** $(1, 1)$: $y = -\frac{1}{2}x + \frac{3}{2}$; **at** $x = 0$: $Q = \left(0, \frac{3}{2}\right)$

REFERENCE GRAPH: $f(x) = x^2 + 3x$ **WITH TANGENT & NORMAL AT** $(1, 4)$

